

Local estimates for entropy densities in coupled map lattices

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We present a method to derive an upper bound for the entropy density of coupled map lattices with local interactions from local observations. To do this, we use an embedding technique being a combination of time delay and spatial embedding. This embedding allows us to identify the local character of the equations of motion. Based on this method we present an approximate estimate of the entropy density by the correlation integral.

Many natural phenomena involving a huge number of degrees of freedom emerge from spatially extended systems (SESs), with translationally invariant equations of motion and spatial coupling. Thus an understanding of these kinds of systems is vital for the understanding of nature. Only for very few of them there exist models from first principles which allow us to explore them by analytical or numerical means. Many systems can not be satisfactorily modeled, so that one has to rely on the study of experimental observations. This usually leads to time series analysis, which has gained new stimulus in recent years through the concept of phase space reconstruction. This approach is quite successful for low dimensional chaotic systems [1,2]. There one can obtain characteristic quantities like Lyapunov exponents, fractal dimensions and entropies directly from the measurements. In some cases it is even possible to construct reasonable model equations from the data.

Natural SESs are typically continuous in time and space and thus described by partial differential equations (PDEs). Nevertheless, a PDE can be approximated by a coupled map lattice (CML), i.e. a model discrete in time and space. CMLs were used as paradigmatic models to study general properties of spatio-temporal chaotic systems. A fundamental question concerning SESs is whether it is possible to extract characteristic quantities from local measurements. This problem is still unsolved in the realm of time series analysis. There were a considerable number of attempts, especially for the dimension density [3–9]. Nevertheless, the outcome was far from being satisfactory. Common to all these works is to use either time delay or purely spatial embedding for reconstructing the dynamical states or the invariant measure. The general problem is that local measurements can only reconstruct finite dimensional subspaces, though the whole phase space is infinite dimensional, in principle. Therefore, these subsystems are never fully deterministic, since they are coupled to the unobserved part of the phase space and are thus open systems.

However, in systems with local, e.g. nearest neighbour, interactions, the equations of motion include only a few local variables. Regarding only these variables the “local” future is governed by deterministic laws, while the whole dynamics in the measured subspace remains “stochastically” driven. The main idea we want to present is that the local deterministic structure can be exploited to estimate a dynamical entropy which will turn out to be a good approximation for the entropy density of the whole (mainly unobserved) system.

The general CML we want to study here is given by

$$x_i(n+1) = (1 - \sigma_r - \sigma_l)f(x_i(n)) + \sigma_l g(x_{i-1}(n)) + \sigma_r g(x_{i+1}(n)), \quad (1)$$

where i represents the position in space and n the time. σ_r and σ_l are spatial coupling constants. Concerning the coupling, typically two kinds of systems are studied: (1) $\sigma_r = \sigma_l = \sigma$ which we will refer to as the (symmetric) diffusive coupling and (2) $\sigma_{r(l)} = 0$, $\sigma_{l(r)} = \sigma$ the unidirectional coupling. The map f is a chaotic map which describes the local dynamics. We will use the tent map $f(x) = 1 - 2|x - 1/2|$ throughout this paper. g is in principle arbitrary function. We use either $g = f$ which gives rise to a nonlinear coupling or $g(x) = x$ which results in a linear coupling.

It is commonly observed that the dynamics of these CMLs is extensively chaotic in suitable parameter regions. Extensive means that the attractor dimension and the KS-entropy are proportional to the system size, here the number N of lattice sites. Thus it is possible to define dimension and entropy densities as intensive quantities. Usually these densities are estimated via the Pesin identity or the Kaplan-York formula, respectively, by calculating the Lyapunov spectrum using the model equations. In some cases this is even possible if the equations of motion are unknown [10].

In previous attempts to calculate entropies and dimensions from observed time series either pure spatial embedding with state vectors of the form

$(x_i(n), \dots, x_{i+l-1}(n))$ or temporal embedding with the vectors $(x_i(n), \dots, x_i(n+m-1))$ were used. Since the dynamics for all these embeddings is non-autonomous with respect to equation (1), the corresponding projections of the invariant measure of the CML contain stochastic components so that the entropy diverges [11]. In this letter we introduce a novel embedding procedure (“pyramid embedding”) which does not increase the dimension of the reconstructed measure when the dimension of the involved subspace is increased. This will enable us to compute the finite entropy related to the deterministic part of the dynamics in the subspace.

Consider a partitioning of a subspace $\Gamma_{\vec{s}}$ spanned by the components of a state vector \vec{s} with a rectangular grid of mesh size ϵ . Via the invariant measure we can assign a probability p_i to every cell and define the entropy

$$H(\vec{s}, \epsilon) = - \sum_i p_i \ln p_i . \quad (2)$$

For sufficient small ϵ its ϵ -dependence is given by [11]

$$H(\vec{s}, \epsilon) \propto -D \ln \epsilon , \quad (3)$$

where D is the information dimension of the projection of the invariant measure into the subspace spanned by \vec{s} . Let us choose a second subspace $\Gamma_{\vec{t}}$ spanned by the components of another state vector \vec{t} satisfying

$$\vec{t} = \vec{F}(\vec{s}) , \quad (4)$$

which implies that the constraint \vec{F} is determined by equation (1). $H(\vec{t}, \vec{s}, \epsilon)$ denotes the entropy of the joint probability p_{ij} for the system being in cell i of $\Gamma_{\vec{s}}$ and in cell j of $\Gamma_{\vec{t}}$. The projection of the invariant measure in the enlarged state space has the identical information dimension D due to the constraints (4). Therefore the conditional entropy

$$h(\vec{t}|\vec{s}, \epsilon) = H(\vec{t}, \vec{s}, \epsilon) - H(\vec{s}, \epsilon) \quad (5)$$

will become independent of ϵ for sufficient small ϵ .

In the following we will use a abbreviated symbolic representation for the states appearing as arguments in the entropies. The pure spatial state of l neighbouring sites will be denoted by

$$(x_1(n), x_2(n), \dots, x_{l-1}(n), x_l(n)) \rightarrow \underbrace{\square \cdots \square}_l .$$

This notation will allow us to write states which combine spatial and temporal embedding in a compact way. Note that we can omit the time and space indices because of the stationarity and translation invariance.

The simplest way to choose state vectors \vec{s} and \vec{t} fulfilling the constraint (4) is $\vec{s} = (x_i(n), x_{i+1}(n))$ and $\vec{t} = (x_i(n+1))$ for unidirectional coupling and $\vec{s} =$

$(x_{i-1}(n), x_i(n), x_{i+1}(n))$ and $\vec{t} = (x_i(n+1))$ for diffusive coupling, respectively. Using the symbolic writing we get for the unidirectional case

$$H(\vec{s}, \epsilon) =: H(\underbrace{\square \square}_2) \quad \text{and} \quad H(\vec{t}, \vec{s}, \epsilon) =: H\left(\underbrace{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}_2\right) . \quad (6)$$

In the symbolic notation of conditional entropies $h(\vec{t}|\vec{s})$ we will hatch \vec{t} , e.g. (6) yields

$$h(\vec{t}|\vec{s}, \epsilon) = H(\vec{t}, \vec{s}, \epsilon) - H(\vec{s}, \epsilon) =: h\left(\underbrace{\begin{smallmatrix} \text{hatched} & \square \\ \square & \square \end{smallmatrix}}_2\right) .$$

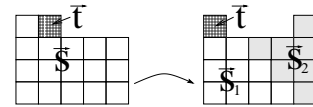
The KS-entropy of a dynamical system is defined as the conditional entropy of the state of the system knowing the full past. Because the KS-entropy is proportional to the system size N in our case, we can introduce the entropy density η which is the KS-entropy divided by N . The definition of the entropy density can be written as

$$\eta = \lim_{N \rightarrow \infty} \lim_{m \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \frac{1}{N} \left(h \left(\underbrace{\begin{smallmatrix} \text{hatched} & \cdots & \text{hatched} \\ \square & \cdots & \square \end{smallmatrix}}_{m+1} \right) \right) , \quad (7)$$

where the entropies are calculated for a finite CML with N lattice sites. We rewrite the r.h.s. of (7) in such a way that only a single site remains as the conditioned part in the conditional entropies [12]:

$$h \left(\underbrace{\begin{smallmatrix} \text{hatched} & \cdots & \text{hatched} \\ \square & \cdots & \square \end{smallmatrix}}_N \right) = h \left(\underbrace{\begin{smallmatrix} \text{hatched} & \cdots & \square \\ \square & \cdots & \square \end{smallmatrix}}_N \right) + \dots + h \left(\underbrace{\begin{smallmatrix} \square & \cdots & \text{hatched} \\ \square & \cdots & \square \end{smallmatrix}}_N \right) \quad (8)$$

The translational invariance and periodic boundary conditions allow cyclic permutations of columns. After shifting \vec{t} to the leftmost column in each single term in (8), we split the blocks \vec{s} into \vec{s}_1 and \vec{s}_2 like



For any \vec{s}_2 , $h(\vec{t}|\vec{s}_1 \vec{s}_2)$ fulfills the inequality

$$h(\vec{t}|\vec{s}_1) \geq h(\vec{t}|\vec{s}_1 \vec{s}_2) \quad (9)$$

It says that the uncertainty about the state \vec{t} is the larger the less I know about the rest of the system. Formally this can be shown using Jensen's inequality (see e.g. [13]). If we apply (9) to (8) we get

$$h \left(\underbrace{\begin{smallmatrix} \text{hatched} & \cdots & \text{hatched} \\ \square & \cdots & \square \end{smallmatrix}}_{m+1} \right) \leq N h \left(\underbrace{\begin{smallmatrix} \text{hatched} & \cdots & \square \\ \square & \cdots & \square \end{smallmatrix}}_{m+1} \right) . \quad (10)$$

For the sake of clarity we restricted the argumentation to the unidirectional case. It will be obvious how the same reasoning can be applied to the diffusive case.

Let us introduce the abbreviation

$$h_p(m, 1) := h \left(m+1 \left\{ \begin{array}{c} \blacksquare \\ \blacksquare \blacksquare \\ \blacksquare \blacksquare \blacksquare \\ \dots \end{array} \right\} \right). \quad (11)$$

The index p means "pyramid" and denotes the form of

$$h_p(2, 3) = h \left(\begin{array}{c} \blacksquare \blacksquare \blacksquare \\ \blacksquare \blacksquare \blacksquare \\ \blacksquare \blacksquare \blacksquare \end{array} \right).$$

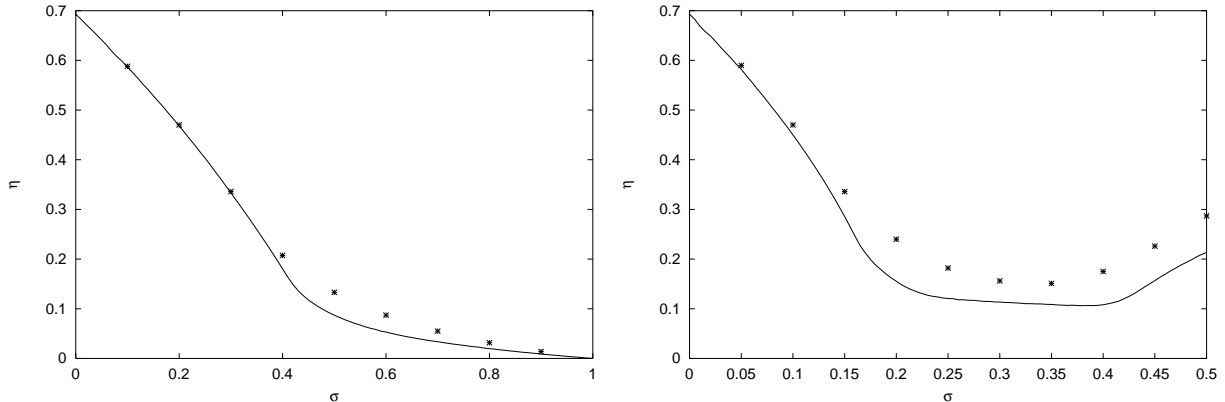


FIG. 1. Estimates of $h_p(1, 1)$ (stars) for the unidirectional linear coupling (left) and the diffusive nonlinear one (right) compared to the entropy density computed via the Pesin identity, η_λ .

Now we can formulate an upper bound for the entropy density:

$$\eta \leq \lim_{\epsilon \rightarrow 0} \frac{1}{n} h_p(m, n). \quad (12)$$

Here we used again the same type of inequality like (9) which leads to

$$h_p(m, n) \leq h_p(m', n) \quad \text{if} \quad m > m'. \quad (13)$$

$$\frac{1}{n} h_p(m, n) \leq \frac{1}{n'} h_p(m, n') \quad \text{if} \quad n > n'. \quad (14)$$

Similar to the usual conditional entropies in low dimensional systems the entropies $h_p(m, n)$ become constant on sufficiently small length scales (see e.g. Fig. 2). Thus the entropies $h_p(m, n)$ provide an upper bound for the entropy density already on finite length scales ϵ . If one is interested in the quality of the bound one has to consider the inequalities (13), which reflect the effects of correlations in time, and (14), which reflect the spatial correlations. Generally, the tighter the bound (9) should be, the less local the measurement has to be.

While the results in the former section were derived for the usual Shannon entropy, for numerical investigations the entropies based on the correlation sum are much more convenient. They provide better statistics and require less computational effort. The correlation sum is defined as

$$C(\epsilon) = \frac{1}{N(N-1)} \sum_{i \neq j} \Theta(\epsilon - |\vec{s}_i - \vec{s}_j|). \quad (15)$$

the spatio-temporal embedding. The first argument m gives the number of time steps used for prediction and the second argument denotes the number of lattice sites predicted. That means e.g.

The quantity $H(\vec{s}, \epsilon) = -\ln C(\epsilon)$ can be regarded as a generalized entropy, the so called correlation entropy [14–16]. One disadvantage of using the correlation entropy is that inequality (9) does no longer hold rigorously. Although experience shows that the deviations are usually small, to our knowledge there do not exist theoretical arguments supporting this. A second disadvantage is that the correlation entropy is a lower bound of the Shannon entropy. Thus, strictly speaking we cannot expect to estimate an upper bound of the entropy density by using the correlation integral. Nevertheless, we can interpret the results provided by the correlation sum as approximate estimates of the entropy density as it was done with the usual correlation entropies as approximate estimates for the KS-entropies of low dimensional systems (e.g. [14]).

Fig. 1 shows the estimates of $h_p(1, 1)$ as a function of the coupling σ estimated by using the correlation sum (stars). They are compared to the results for the entropy density calculated by the Pesin identity (solid line) via the Lyapunov exponents $\eta_\lambda = 1/N \sum_i \lambda_i$ with $\lambda_i > 0$. The scaling with respect to ϵ can be seen in Fig. 2 for the unidirectional coupling with $\sigma = 0.2$ and $\sigma = 0.5$. In the example with $\sigma = 0.2$ the estimates of $h_p(m, 1)$ are almost independent of m and turn out to be a good approximation of the entropy density, calculated by the Pesin-identity. This coincides with the observation in Fig. 1 that the value of $h_p(1, 1)$ is very close to the value

calculated via the Lyapunov exponents for $\sigma \leq 0.3$ for the unidirectional and $\sigma \leq 0.05$ in the diffusive case. For larger coupling effects of correlations become visible. In Fig. 2 one sees that $h_p(2, 1)$ is a remarkably better esti-

mate than $h_p(1, 1)$. Further increasing m gives no better results. The remaining difference between $h_p(m, 1)$ and the Pesin value might be due to the spatial correlations in the system.

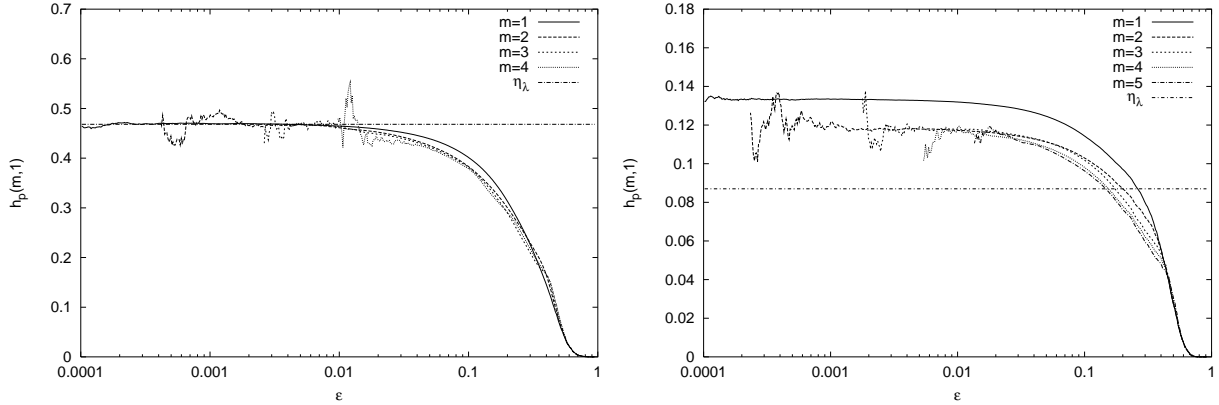


FIG. 2. Estimates of $h_p(m, 1)$ for the unidirectional case, linear coupling with $\sigma = 0.2$ (left) and $\sigma = 0.5$ (right).

Moreover, in the strong coupling case the problem of the violation of (9) by the correlation entropies becomes relevant. Fig. 3 shows $h_p(m, 1)$ for $m = 1, 2, 3$. As one can see the inequality (13) is strongly violated. The reason for this behaviour is still an open question.

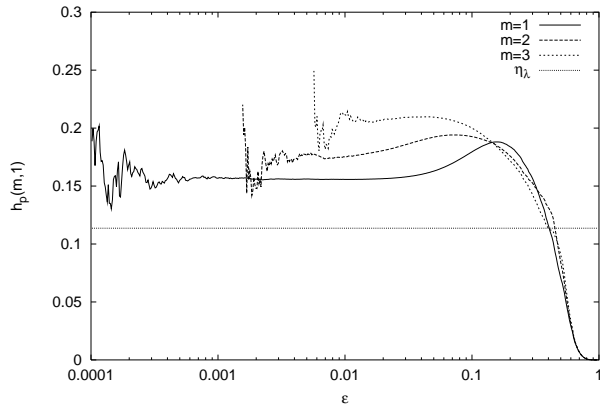


FIG. 3. $h_p(m, 1)$ $m = 1, 2, 3$ for the diffusive case, $\sigma = 0.3$ estimated by using the correlation entropies.

In summary, we presented a method to estimate the entropy density in coupled map lattices with local couplings using only observables of local subsystems which corresponds to the estimates of the KS-entropy in low dimensional systems. As shown in eqn. (7)-(10) this would be an upper bound of the KS-entropy, if we could use an algorithm based on a partition of the phase space. Unfortunately, this is not possible due to the statistical requirements. The correlation method uses a covering, instead, so that the inequalities shown are no longer valid, rigorously. Nevertheless, the numerical investigation we presented for two simple cases showed that this method

provides a rather good approximation of the entropy density.

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